SPRING 2025 MATH 590: QUIZ 11

Name:

1. Find the JCF and the change of basis matrix for $A = \begin{pmatrix} 0 & 25 \\ -1 & 10 \end{pmatrix}$. (5 points)

Solution. $p_A(x) = \begin{vmatrix} x & -25 \\ 1 & x-10 \end{vmatrix} = x^2 - 10x + 25 = (x-5)^2. E_5 =$ null space $\begin{pmatrix} -5 & 25 \\ -1 & 5 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & -5 \\ 0 & 0 \end{pmatrix}$. Thus, $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ is a basis for E_5, so the JCF is $\begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$.

To find the change of basis matrix, take v_2, any vector not in E_5, say $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then we take $v_1 = \begin{pmatrix} -5 & 25 \\ -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$, so that $P = \begin{pmatrix} -5 & 1 \\ -1 & 0 \end{pmatrix}$.

2. Follow the steps below to find the JCF and the corresponding change of basis matrix for $B = \begin{pmatrix} 4 & 0 & -2 \\ 1 & 2 & -1 \\ 2 & 0 & 0 \end{pmatrix}$.

(i) Find $p_A(x)$ and the single eigenvalue λ.
(ii) Calculate E_λ.
(iii) Write down the JCF J.
(iv) Find $v_2 \notin E_\lambda$.
(v) Set $v_1 := (A - \lambda I)v_2$. This turns out to be a vector in E_λ.
(vi) Take $v_3 \in E_\lambda$ not a multiple of v_1.
(vii) Letting P be the matrix whose columns are v_1, v_2, v_3, verify that $P^{-1}AP = J$. (Hint': You don't have to find P^{-1} to do this.)

Solution. (i) $p_A(x) = \begin{vmatrix} x-4 & 0 & 2 \\ -1 & x-2 & 1 \\ -2 & 0 & x \end{vmatrix} = -2(-2x+4) + x((x-4)(x-2)) = (x-2)\{x^2 - 4x + 4\} = (x-2)^3$. Thus, 2 is the only eigenvalue.

(ii) E_2 is the nullspace of $\begin{pmatrix} 2 & 0 & -2 \\ 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so that $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ form a basis for E_2.

(iii) The JCF is $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, since the number of Jordan blocks is two.

(iv) Take $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(v) $v_1 = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

(vi) We can take $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, so that $P = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$.

(vii) $AP = \begin{pmatrix} 4 & 0 & -2 \\ 1 & 2 & -1 \\ 2 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 0 \\ 2 & 1 & 2 \\ 4 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = PJ$, so $P^{-1}AP = J$.